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Spectral invariants for coupled spin-oscillators

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This text deals with *inverse spectral theory* in a semiclassical setting. Given a quantum system, the haunting question is « What interesting quantities can be discovered on the spectrum that can help to characterize the system ? » The general framework will be semiclassical analysis, and the issue is to recover the classical dynamics from the quantum spectrum. The coupling of a spin and an oscillator is a fundamental example in physics where some nontrivial explicit calculations can be done.

1 Inverse spectral problems

Inverse spectral problems are well known and broadly studied in mathematics. They have obvious applications in physics, biology, chemistry, etc.

The most famous problem for a mathematician was advertized by Kac [14] : *can one hear the shape of a drum ?* It concerns the discrete spectrum of the Laplacian Δ on a compact Euclidean domain $\Omega \in \mathbb{R}^n$, with some boundary conditions (usually Dirichlet).

Question: determine Ω from the spectrum of Δ... up to isometry. The isometry group is of course very important. It will be crucial in our setting below.

An old variant of this question concerns the Laplace-Beltrami Δ on a compact Riemannian manifold (M, g) :

Question: determine the metric g from the spectrum of Δ.

As is well known, the simple answer to both questions is: no! The riemannian case was settled by Milnor in 1964 [15], who showed that « there exist two Riemannian flat tori of dimension 16 which are not globally isometric (though affinely equivalent), and yet whose Laplacian for exterior forms has the same sequence of eigenvalues. » [MathSciNet]

The case of a compact domain in $\mathbb{R}^2$ is a bit more recent. In 1992, Gordon, Webb and Wolpert [10] exhibited non-isometric polygonal domains with the same spectrum (Figure 1).

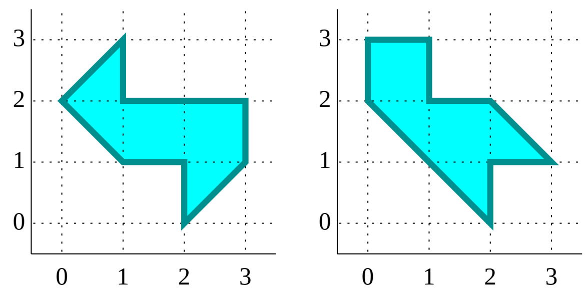

Figure 1: An example of non-isometric polygonal domains with the same spectrum, found by Gordon, Webb, Wolpert.

2 More interesting questions...

These negative results should not lead us to think that subject is over. It is far from over, the true question being: ...what do we really understand ?

To make my point, here is a simple question for which we *don't* have any answer yet:

Open problem: What about when $\Omega \subset \mathbb{R}^2$ is convex ?

In other words, are there two non-isometric convex domains that give the same spectrum ? The situation is even worse, as the following is still open :

Open problem: What about when $\Omega \subset \mathbb{R}^2$ with $\partial\Omega$ analytic ?

The most recent advances on these question are due to works by Zelditch 1995–2009, see [28, 27, 29]. Here is what Steve Zelditch says on his webpage about the last article :

> « Second paper in a series on the inverse spectral problem for analytic plane domains. In the first paper in the series, I give rigorous version of the Balian-Bloch trace asymptotics for the resolvent. It gives a new algorithm for calculating wave invariants associated to periodic reflecting rays of bounded domains. In this paper, I calculate the wave invariants for a bouncing ball orbit (among others), and use them to determine the domain when the domain is analytic and possesses one symmetry. The symmetry is assumed to interchange the two endpoints of the bouncing ball orbit. The calculation uses some new methods and I have spent a fair amount of effort verifying the different steps before submitting the article. Any comments would be much appreciated. »

It is worth noticing that many methods involved are *not* specific to the

Laplacian. Normal forms, microlocal analysis... these suggest that one could do something for more general operators.

3 Semiclassical Schrödinger

The inverse spectral theory for the Schrödinger operator is also an old subject. Here we assume that the potential V is C^∞.
Schrödinger operator on $L^2(\mathbb{R}^n)$: $-\frac{\hbar^2}{2}\Delta + V(x)$.

An important feature of this equation is the semiclassical parameter $\hbar$. Of course, if V is zero, we recover the Laplacian, but the semiclassical parameter induces a slight change of viewpoint: we shall focus now only on a spectral window close to some given energy, but in the asymptotic regime $\hbar \to 0$. For the Laplacian, this amounts to looking only at the high energy spectrum.

Naturally, the inverse question is the following :
Question: Can we recover the potential from the semiclassical spectrum ?

Precisely, this means that we shall need the spectrum not only for a fixed value of $\hbar$, but for a whole sequence of values of $\hbar$ that should accumulate at zero. Hence the phrase "semiclassical spectrum". It can be misleading to some people : it is not a "spectrum" computed from classical data, but an infinite collection of true quantum mechanical spectra.

Again, semiclassical inverse spectral theory is not new, but it is clearly of renewed interest. Several people are currently very active on related questions [11], [13], [12], etc.

I would like to mention two recent advances in the particular case of $1D$ Schrödinger operators with smooth potential.

Theorem 3.1 (Colin de Verdière – Guillemin [5]) *The* Taylor expansion *of the potential at a generic non-degenerate critical point is determined by the semiclassical spectrum of the associated Schrödinger operator near the corresponding critical value.*

Theorem 3.2 (Colin de Verdière [3]) *Under some genericity assumption, one can explicitly reconstruct the* full potential *from the semiclassical spectrum.*

I will discuss below the genericity condition of the last theorem. It is a very weak (thus very generic) condition.

4 More general operators

We can ask the inverse spectral question for even more general operators. The Laplacian $\hbar^2\Delta$ and the Schrödinger operator $-\frac{\hbar^2}{2}\Delta+V(x)$ are instances of *semiclassical (pseudo)differential operators*.

Goal: develop an inverse spectral theory for general semiclassical operators.

The subject is not completely new. See for instance [13].

What to recover from the spectrum ? We claim that the only natural classical object that should be recovered from the spectrum is the principal symbol of the pseudodifferential operator. This is a direct generalization of the riemannian case, where the principal symbol clearly encodes the metric g.

However, heuristically, it is not reasonable to try and recover the full principal symbol : it is a function of $2n$ variables, whereas « the shape of the drum », the metric, or the potential only depend on n variables.

In fact, if one works modulo *symplectomorphisms* then determining the principal symbol sounds much more realistic. Indeed, the group of symplectomorphisms can be quite large (larger than the group of isometries of the position space, since any isometry extends to a canonical transformation in phase space, but there are lots of canonical transformations that do not arise from isometries).

Remark: From a more concrete point of view, what does it means to know the principal symbol p only modulo symplectomorphism ? It is clear that if we view p as a classical hamiltonian, giving rise to a hamiltonian vector field $\mathcal{X}_p$, then the *flow* of this vector field is invariant under symplectomorphism.

Thus, the classical dynamics induced by p is the most natural symplectic invariant of the system. This is what we want to recover from the spectrum. With this in mind, our question now becomes : what tractable quantities can be discovered on the spectrum that more or less characterize the dynamics ?

5 Inverse result for Morse functions in 1D

I would like to explain here a recent result of mine [22] which gives a fairly complete answer in the case of 1D Morse hamiltonians. The setting is as follows.

- $P = P(\hbar)$ is a selfadjoint pseudodifferential operator in $\Psi^0(m)$ with principal symbol p, elliptic at infinity.
- $\hbar \in \mathcal{J}$ where $\mathcal{J} \subset [0,1]$ is an infinite subset with zero as an accumulation point.

- There exists a neighbourhood J of I such that $p^{-1}(J)$ is compact in M.

Recall that in the semiclassical theory, the ellipticity at infinity means that $|p(x,\xi)| \geqslant \frac{1}{C}(|x|^2 + |\xi|^2)^{m/2}$ for $|x|^2 + |\xi|^2 \geqslant C$.

The following is well known.

Proposition 5.1 *Under these hypothesis, for any open interval $I \subset J$ there exists $\hbar_0 > 0$ such that the spectrum of P in I is* ***discrete*** *for $\hbar \leqslant \hbar_0$.*

We concentrate on the spectrum that lies inside the interval I. Thus, our phase space is now $M = p^{-1}(I)$.

Theorem 5.2 ([22]) *Suppose that $p_{\restriction M}$ is a Morse function. Assume that the graphs of the periods of all trajectories of the hamiltonian flow defined by $p_{\restriction M}$, as functions of the energy, intersect generically.*

Then the knowledge of the spectrum $\Sigma(P, \mathcal{J}, I) + \mathcal{O}(\hbar^2)$ determines the dynamics of the hamiltonian system $p_{\restriction M}$.

More precisely, the generic assumption, similar to the one used in [3], is that any two smooth lagrangian (=curves) in (E, τ) space, where τ is a period of the hamiltonian flow of p at energy $p = E$, should have a non-flat intersection : the contact should have a finite order. It is a very weak condition, but it does rule out, for instance, systems with symmetry (like the symmetric double well potential). For these systems, one could certainly do a similar analysis in the restricted class of symmetric hamiltonians. On the other hand, detecting the symmetry from the spectrum would require other ideas, and might be very interesting.

I find the proof of this result very entertaining. It includes singularity theory, topology, symplectic invariants..., and uses microlocal analysis in phase space and in "time-energy" space. As always in these inverse results, there are two steps : a quantum step and a classical step. The quantum step extracts some numerical invariants from the spectrum; the classical step should prove that these numerical invariants are enough to symplectically discriminate. Here the final ingredient (classical step) is provided by a purely symplectic classification theorem due to Dufour-Molino-Toulet [7].

6 Classical and quantum integrable systems

Of course, one would like to explore higher dimensions. Several directions of study could be natural. I believe that similar methods should be usable under some hypothesis of *complete integrability.* Let me explain this briefly.

Let M be a symplectic manifold of dimension $2n$. A *classical (or Liouville) integrable system* is the data of n functions $p_1, \ldots, p_n$ on M which Poisson-commute : $\{p_i, p_j\} = 0$, and whose differential are almost everywhere independent.

Let $\mathcal{H}_\hbar$ be a Hilbert space quantizing M. A *quantum integrable system* is the data of n self-adjoint semiclassical operators $P_1, \ldots, P_n$ acting on $\mathcal{H}_\hbar$ which commute : $[P_i, P_j] = 0$, and whose principal symbols $p_1, \ldots, p_n$ form a Liouville integrable system on M.

The statement above does not mention then quantization scheme explicitly; indeed, it is natural to expect that the methods should apply not only to pseudodifferential operators, but also to semiclassical Toeplitz operators on compact phase spaces [2].

7 Inverse spectral theory for integrable systems

Since we have a set of commuting operators, instead of a single spectrum one considers the *joint spectrum* of $P_1, \ldots, P_n$ (which we assume to be discrete.)

$$\mathit{Jspec}(P_1, \ldots, P_n) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n, \quad \cap_j \ker(P_j - \lambda_j) \neq 0\}.$$

Naturally, the classical analogue of the joint spectrum is the image of the "joint classical energy map" — usually called "momentum map" or "energy-momentum map", which is the map $F := (p_1, \ldots, p_n) : M \to \mathbb{R}^n$. It is customary, in this setting, to simply call F "the integrable system". Thus, our inverse spectral question can be phrased as follows :
Question: Does the joint spectrum determine the Liouville integrable system $p_1, \ldots, p_n$? (up to symplectic equivalence, of course).

There is no general answer for the moment, as work is in progress, and the problem is vast. With Pelayo, we wrote an attempt to set up a whole research programme in this direction in [16]. In the present text, it is probably wiser (and quicker) to start with examples. For the moment, much of the research has been done in two degrees of freedom, so we will examine three famous 2D examples :

- Harmonic oscillators.
- The quantum spherical pendulum.
- The quantum spin-oscillator coupling.

8 Harmonic oscillators

Two uncoupled harmonic oscillators form one of the simplest quantum integrable systems. All computations are explicit, and it is already interesting to have in mind the structure of the joint spectrum, as it serves as a model for the so-called elliptic-elliptic singularity.

Thus we have two self-adjoint operators acting on $L^2(\mathbb{R}^2)$:

$$P_1 = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x_1^2} + \frac{x_1^2}{2}, \qquad P_2 = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x_2^2} + \frac{x_2^2}{2}.$$

We know that Spectrum$(P_j) = \hbar(\frac{1}{2} + \mathbb{N})$. Therefore, the joint spectrum is as follows (see Figure 2) :

$$Jspec(P_1, P_2) = \hbar\left((\frac{1}{2}, \frac{1}{2}) + \mathbb{N}^2 \right).$$

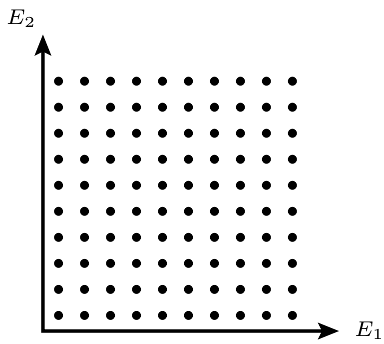

Figure 2: Joint spectrum of harmonic oscillators. It extends indefinitely in the positive quadrant.

From a geometric viewpoint, the key feature of such a simple system is that the classical symbols $p_1 = (\xi_1^2 + x_1^2)/2$ and $p_2 = (\xi_2^2 + x_2^2)/2$ generate a hamiltonian 2-torus action on $\mathbb{R}^4$. It was proved by Colin de Verdière [4] that each time you have a completely integrable system whose symbols define a torus action, then the joint spectrum is very similar to Figure 2 : it is the intersection of a $\hbar$-lattice with a convex cone.

9 Spherical pendulum

The spherical pendulum, an ancient system already studied by Huygens, is now tightly associated with Hans Duistermaat and Richard Cushman [8, 6], who discovered a non-trivial topological invariant (the monodromy) that, contrary to the case of harmonic oscillators, prevents the system from being a hamiltonian torus action. I proved in [23] that this invariant can be detected on the joint spectrum (see Figure 3). The two commuting operators are acting on $L^2(S^2)$:

$$\hat{H} = -\tfrac{\hbar^2}{2}\Delta + z, \qquad \hat{J} = \frac{\hbar}{i}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) = \frac{\hbar}{i}\frac{\partial}{\partial \theta}.$$

There is no explicit formula for the spectrum. The spectrum in Figure 3) was computed by numerical diagonalisation after compression on a sufficiently large orthonormal basis.

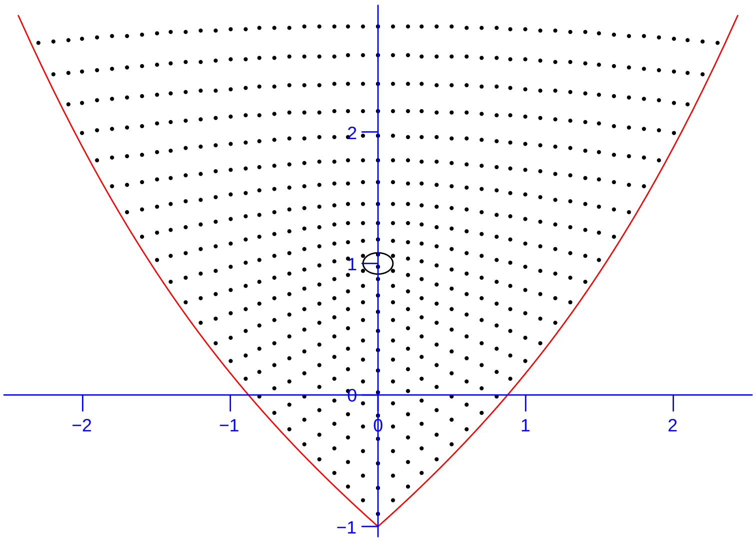

Figure 3: Joint spectrum of the Spherical pendulum. The peculiar arrangement of eigenvalues around the point $(0, 1)$ is a characterization of quantum monodromy.

10 Classical Spin-oscillator coupling

Our last example has a strong physics flavour. It is naturally obtained when one studies the coupling of a quantum spin with a simple harmonic oscillator. This can be used, for instance, to confine a quantum particle with a finite number of states within a quantum well. In the physics literature, it is called the Jaynes-Cummings model [1]. What follows is a joint work with Á. Pelayo [18].

It is easier to introduce first the classical version. The phase space is $M = S^2_{(x,y,z)} \times \mathbb{R}^2_{(u,v)}$, and the commuting hamiltonians are

$$J := (u^2 + v^2)/2 + z, \qquad H := \frac{1}{2}\,(ux + vy).$$

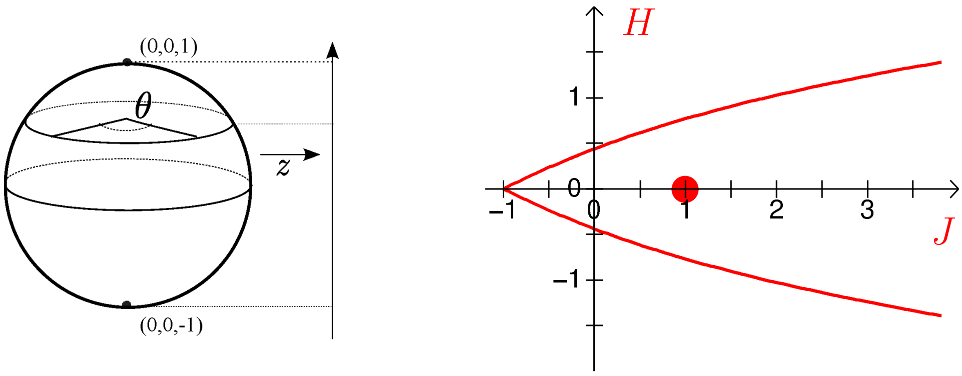

Figure 4: The figure on the right shows the bifurcation diagram of the classical spin–oscillator system.

A straightforward singularity computation gives the following (see for instance the survey [20] for the definitions) :

Proposition 10.1 ([18]) *The singularities of the coupled spin–oscillator are non-degenerate and of elliptic-elliptic, transversally-elliptic or focus-focus type* (it is a semitoric system)*. It has exactly one focus-focus singularity at the "North Pole"* $((0,\,0,\,1),\,(0,\,0)) \in S^2 \times \mathbb{R}^2$ *and one elliptic-elliptic singularity at the "South Pole"* $((0,\,0,\,-1),\,(0,\,0))$*.*

The Quantum sphere (spin) Since the phase space of the spin–oscillator is not a cotangent bundle, it cannot be quantised by ordinary (pseudo)-differential operators. However, the sphere S^2 has a very simple and natural semiclassical quantization, which we describe now.

Let $L(z_1,\, z_2) = \frac{|z_1|^2+|z_2|^2}{2}$. This harmonic oscillator generates a Hamiltonian S^1-action on $\mathbb{C}^2 = \mathbb{R}^4$. The level set $L^{-1}(2)$ is a 3-sphere. We use the

Hopf fibration $S^3_2 \to S^2$ where $S^3_2 := L^{-1}(2) \subset \mathbb{C}^2$, given by

$$\begin{aligned} x &= \mathrm{Re}(z_1\bar{z}_2)/2 \\ y &= \mathrm{Im}(z_1\bar{z}_2)/2 \\ z &= (|z_1|^2 - |z_2|^2)/4. \end{aligned}$$

Then we have $S^2 = S^3_2/S^1$. Thus S^2 is the so-called *Marsden-Weinstein* reduction of $\mathbb{C}^2$ by this hamiltonian S^1-action. We use the idea of *Quantization commutes with reduction*: the quantum Hilbert space associated to S^2 should be the eigenspace of $\hat{L}$ corresponding to the classical energy $L = 2$. Precisely, the quantization of $\mathbb{R}^4$ is $L^2(\mathbb{R}^2)$, the quantization of L is $\hat{L} = -\frac{\hbar^2}{2}\left(\frac{d^2}{dx_1^2} + \frac{d^2}{dx_2^2}\right) + \frac{x_1^2+x_2^2}{2}$. Thus we define the quantization of S^2 to be $\mathcal{H} := \ker(\hat{L} - 2)$.

Notice that $\mathcal{H}$ is a finite dimensional Hilbert space. If $2 = \hbar(n+1)$, then $\dim(\mathcal{H}) = n+1$ (otherwise $\mathcal{H} = \{0\}$).

Quantum spin-oscillator We may quantize the Hopf map in a natural way (Weyl quantization) by viewing it as a map defined on $\mathbb{R}^4$. Then, we may define the quantization of J and H :

Definition 10.2 *The* quantization of J *is the operator*

$$\hat{J} = Id \otimes \Big(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial u^2} + \frac{u^2}{2} \Big) + (\hat{z} \otimes Id).$$

The quantization of H *is the operator*

$$\hat{H} = \frac{1}{2}(\hat{x} \otimes u + \hat{y} \otimes (\frac{\hbar}{i}\frac{\partial}{\partial u})),$$

both acting on the Hilbert space $\mathcal{H} \otimes L^2(\mathbb{R})$*.*

With these definitions, the *joint spectrum* is easy to compute numerically, since we have an explicit expression for the matrix of the operators $\hat{J}$ and $\hat{H}$ in a suitable basis.

Proposition 10.3 *For each eigenvalue λ of $\hat{J}$, $\lambda = \hbar(\frac{1-n}{2} + \ell_0)$, $\ell_0 \in \mathbb{N}$, there is a basis of* $\ker \hat{J} - \lambda$ *in which the matrix of $\hat{H}$ is*

$$\left(\frac{\hbar}{2}\right)^{\frac{3}{2}} \begin{pmatrix} 0 & \beta_1 & \dots & & & 0 \\ \beta_1 & 0 & \beta_2 & & & 0 \\ 0 & \beta_2 & 0 & \beta_3 & & 0 \\ & & & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & & & & & \beta_\mu \\ 0 & 0 & \dots & & \beta_\mu & 0 \end{pmatrix}, \quad \beta_k := \sqrt{(\ell_0 + 1 - k)k(n - k + 1)}.$$

Spin-Oscillator: the joint spectrum The figure 5 shows the joint spectrum of the spin-oscillator model. Although it looks similar to the spherical

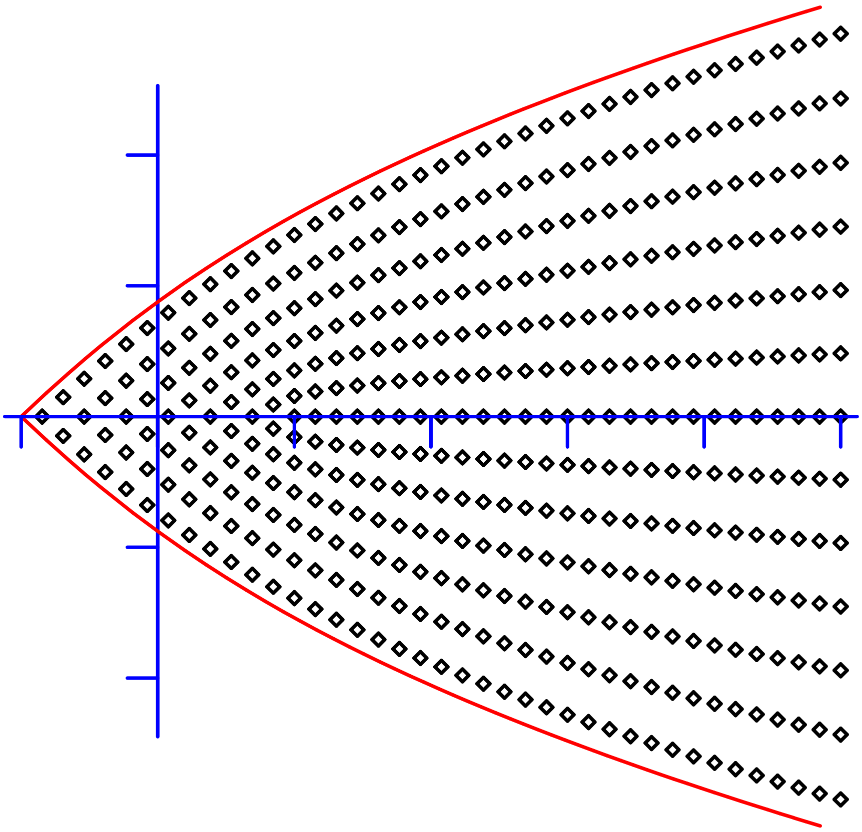

Figure 5: The joint spectrum of the spin-oscillator.

pendulum — and indeed, it demonstrates quantum monodromy as well — it has a distinct feature that makes it easier to study : vertical sections are bounded. The vertical direction here, and in fact in all of our three examples, is special in the sense that the corresponding hamiltonian, J, defines an S^1-action. The quantum manifestation of this fact can be detected : the joint eigenvalues all lie on regularly spaced vertical lines.

A 2D system with such an S^1-action is called *semitoric*. In [17] and [19], we have classified these semitoric systems up to symplectic equivalence. A pioneer study of the joint spectrum of particular examples of such systems was made by Sadovskii and Zhilinskii [21].

We don't know yet whether the spectrum of a semitoric system determines its classical dynamics, but there are a number of facts that support this conjecture. We mentioned earlier that the monodromy can be detected. Other quantities can be, too. For instance, one should be able to prove that the Duistermaat-Heckman measure associated to $J = n\hbar$ should simply be the (rescaled) number of eigenvalues in the vertical line $x = n\hbar$. It is also tempting to try and recover Maslov indices from the "boundary" of the spectrum.

In the remaining of this text we want to give numerical evidence that a *dynamical invariant* associated to the singularity at $(1, 0)$ can be detected from the concentration of eigenvalues. This invariant is one of the combinatorial data that was used in [17] for classifying semitoric systems.

11 Eigenvalue concentration

We start with a numerical exploration related to eigenvalue concentration. This was done in the spin-oscillator example, but is clearly much more universal.

Let $\Sigma(n) = \{E_0 \leq E_1 \leq \ldots \leq E_n\}$ be the spectrum of $\hat{H}|_{\ker(\hat{J}-\mathrm{Id})}$. This is the *focus-focus spectrum*, in the sense that the corresponding value $J = 1$ is precisely the image of the focus-focus point.

Let $t^{\min}(\hbar) = \min_k \left(\frac{E_{k+1}-E_k}{\hbar}\right)$ be the rescaled lowest eigenvalue spacing, as a function of $\hbar$.

Conjecture 11.1 *The following classical limits exist :*

$$B = \lim_{\hbar\to 0} \left(\frac{2\pi}{t^{\min} |\ln \hbar|}\right).$$

$$a = \lim_{\hbar\to 0} \left(\frac{2\pi}{Bt^{\min}} - |\ln \hbar|\right).$$

Notice that if the second limit exists, this means that B has in fact a two-term asymptotic expansion. Assuming this, we get a more accurate "accelerated expression" for B by using two small values of $\hbar$: $\hbar_1$ and $\hbar_2$:

$$B = \lim_{\hbar \to 0} \left(\frac{2\pi}{t^{\min} |\ln \hbar|} \right) \quad = \frac{\frac{2\pi}{t^{\min}(\hbar_1)} - \frac{2\pi}{t^{\min}(\hbar_2)}}{\ln(\hbar_2/\hbar_1)} + \mathcal{O}(\hbar_1 \ln \hbar_1) + \mathcal{O}(\hbar_2 \ln \hbar_2).$$

This formula is very efficient numerically if we choose a fixed ratio $\hbar_1/\hbar_2$. We have plotted these quantities in Figures 6 and 7 which suggest $B \simeq 2$, $a \simeq 4,7$. How can we interpret this ?

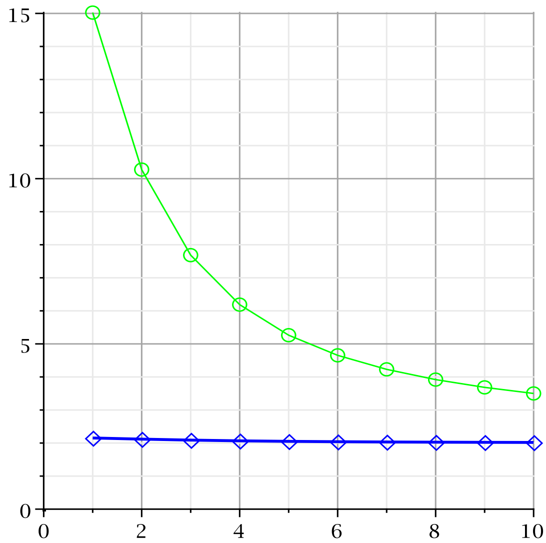

Figure 6: Recovering dynamical invariant B. The abscissa is logarithmic : it is an integer k such that $\hbar = 1/(2^{k-1}+1)$. Thus the classical limit $\hbar \to 0$ is observed for large k. The top curve is the naive computation, the bottom one is an accelerated formula taking into account an assumed asymptotic expansion. They show a good convergence to $B = 2$.

Dynamics on the singular lagrangian manifold $\Lambda_0 = \{J = 1\} \cap \{H = 0\}$. We claim that these quantities should be related to the dynamics of the classical system on the singular lagrangian manifold $\Lambda_0 = \{J = 1\} \cap \{H = 0\}$. Apart from being singular, this manifold is in fact easier to study than the neighboring regular Liouville tori, because it has an explicit parametrization in terms of simple functions.

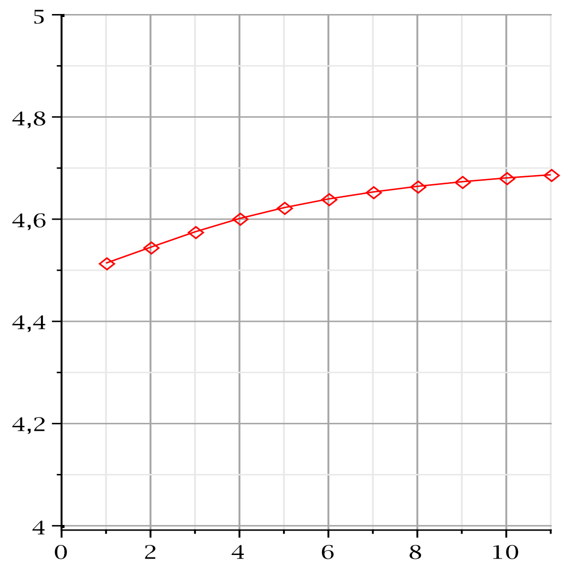

Figure 7: Recovering dynamical invariant a. The abscissa is the same as in Figure 6. We find from this plot $a \simeq 4,7$.

We work in polar coordinates: $u + iv = re^{it}$ and $x + iy = \rho e^{i\theta}$. For $\epsilon = \pm 1$, we consider the mapping $S_\epsilon : [-1,\, 1] \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2 \times S^2$ given by (here $p = (\tilde{z},\, \tilde{\theta}) \in [-1, 1] \times [0, 2\pi)$):

$$\begin{cases} r(p) = \sqrt{2(1-\tilde{z})} \\ t(p) = \tilde{\theta} + \epsilon\frac{\pi}{2} \\ \rho(p) = \sqrt{1-\tilde{z}^2} \\ \theta(p) = \tilde{\theta} \\ z(p) = \tilde{z}. \end{cases}$$

Proposition 11.2 *The map S_ϵ, where $\epsilon = \pm 1$, is continuous and S_ϵ restricted to $(-1,\, 1) \times \mathbb{R}/2\pi\mathbb{Z}$ is a diffeomorphism onto its image. If we let $\Lambda_0^\epsilon := S_\epsilon([-1,\, 1] \times \mathbb{R}/2\pi\mathbb{Z})$, then $\Lambda_0^1 \cup \Lambda_0^2 = \Lambda_0$ and*

$$\Lambda_0^1 \cap \Lambda_0^2 = \Big(\{(0,\, 0)\} \times \{(1,\, 0,\, 0)\}\Big) \cup \Big(C_2 \times \{(0,\, 0,\, -1)\}\Big),$$

where C_2 denotes the circle of radius 2 centered at $(0,\, 0)$ in $\mathbb{R}^2$. Moreover, S_ϵ restricted to $(-1,\, 1) \times \mathbb{R}/2\pi\mathbb{Z}$ is a smooth Lagrangian embedding into $\mathbb{R}^2 \times S^2$.

From this proposition we recover in particular the general fact that Λ_0 is homeomorphic to the embedding of a sphere with a double point, or, equivalently, a "pinched torus" (see Figure 8).

The Taylor series. The general theory of focus-focus singular fibers [25] tells us that the dynamics intrinsically defines a Taylor series in two variables, as described in the following picture (Figure 8). This is a dynamical invariant

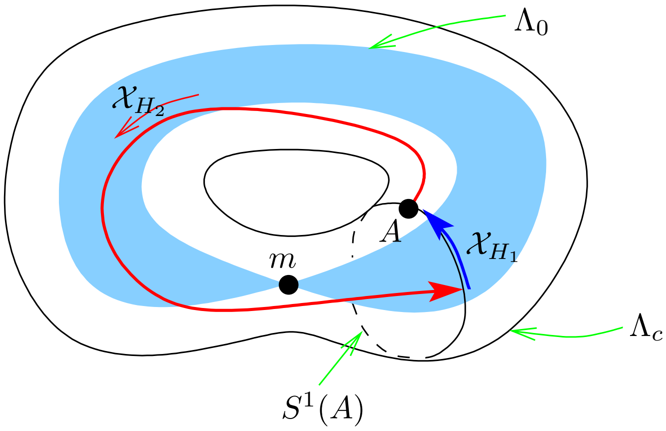

Figure 8: Definition of the Taylor series

associated to a focus-focus singular fiber Λ_0. (Here $\Lambda_0 = \{J = 1\} \cap \{H = 0\}$), defined as follows.

1. Normalize $F = (J, H)$ near Λ_0: replace F by $(H_1, H_2) = g \circ F$ such that near Λ_0, $(H_1, H_2) = (q_1, q_2)$, with $q_1 = x_1\xi_2 - x_2\xi_1$, $q_2 = x_1\xi_1 + x_2\xi_2$, in some local symplectic coordinates near the singular point m (Eliasson's theorem [26].)

2. Notice that the flow of $\mathcal{X}_{H_2}$ is periodic. Let $A \in \Lambda_c := (H_1, H_2)^{-1}(c)$ for some regular value $c \neq 0 \in \mathbb{R}^2$. We flow along the vector field $\mathcal{X}_{H_2}$ until we reach the $\mathcal{X}_{H_2}$-orbit of A again. Then we flow along the vector field $\mathcal{X}_{H_2}$ to go back to A.

3. Let γ_c be the corresponding path (blue + red in the picture) and let $\tau_1(c) + \tau_2(c)$ be the corresponding flow times for $\mathcal{X}_{H_1}$ and $\mathcal{X}_{H_2}$, respectively.

Then the functions $\begin{cases} \sigma_1(c) := & \tau_1(c) - \mathrm{Im}(\ln c) \\ \sigma_2(c) := & \tau_2(c) + \mathrm{Re}(\ln c). \end{cases}$ are smooth (!) and there exists a smooth function $S(c_1, c_2)$ such that $dS = \sigma_1 dc_1 + \sigma_2 dc_2$.

Theorem 11.3 ([25]) *The Taylor series of S characterizes the foliation, up to symplectic equivalence.*

In practice, this Taylor series is quite hard to compute. Very recently, the computation was carried out for the spherical pendulum [9]. In [18], we have simply computed the first terms, which is already quite involved.

Theorem 11.4 ([18]) *For the quantum spin-oscillator coupling, we have*

$$S(X, Y) = \frac{\pi}{2} X + (5 \ln 2) Y + \mathcal{O}((X, Y)^2).$$

This is a purely classical calculation. But from this, we conjecture

Conjecture 11.5 $\lim_{\hbar \to 0} \left(\frac{2\pi}{t^{\min} |\ln \hbar|} \right) = 2.$

$\lim_{\hbar \to 0} \left(\frac{2\pi}{B t^{\min}} - |\ln \hbar| \right) - |\ln \hbar| - \gamma = 5 \ln 2.$

Here γ is Euler's constant, $\gamma \simeq 0.5772156649$. This conjecture (or, in fact, a precised version of Conjecture 11.1 above) is supported by a microlocal formula proved in [24] for pseudodifferential operators. Proving the conjecture would require an adaptation of this formula to the quantization of S^2; this, certainly, should be done using the microlocal theory of semiclassical Toeplitz operators, as developed in [2].

Notice that, in view of Conjecture 11.1, this gives

$$B = 2 \qquad a = 6 \ln 2 + \gamma \simeq 4,74...$$

which is in agreement with the numerical plots.

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